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Emission by multipoles: an exact result in terms of reducible multipoles

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Abstract. On the basis of an expression for the (time-averaged) radiated power comprising the radiation sources at the retarded time, an exact expression relevant to the emission by *reducible* multipoles is obtained, which, along with the emission from independent multipoles, accounts for the emission from the interaction between multipoles of different order. Whereas the former contribution is known in literature, at least for the lower-order multipoles, the latter contribution so far appears to have been overlooked. The result thus obtained is discussed in reference to the treatment of multipole emission in terms of *irreducible* multipoles, generalized to non-monochromatic sources.

1. Introduction

The expansion of electromagnetic quantities in multipoles is a common and useful procedure in electrodynamics [1], particularly in quantum electrodynamics [2, 3]. For time-dependent sources and electromagnetic fields, such an expansion can be based on a description in terms of plane waves, i.e. the sources and the fields are described in terms of their Fourier transforms in both space and time, with the result that Maxwell's equations are reduced to a single algebraic equation, the inhomogeneous wave equation, whose solution is straightforward. Such a procedure can be applied to describe wave processes *in vacuo* as well as in an arbitrary medium [4]. Alternatively, with reference to emission *in vacuo*, one can express the fields, solutions of an Helmholtz-type (homogeneous) wave equation, in a series of vector spherical harmonics, the corresponding expansion coefficients being connected with the sources through appropriate solutions of the inhomogeneous system of Maxwell's equations [1, 2]. The equivalence of the two methods is a consequence of the completeness of the respective sets of basis functions in the Hilbert space $\mathcal{L}^2(\mathbb{R}^3)$ [5]. In this respect, one should note that square integrable solutions are the only ones for which the total electromagnetic energy is finite, so that all physical solutions belong to the Hilbert space $\mathcal{L}^2(\mathbb{R}^3)$.

As for the radiated power in the form of a multipole expansion, the solution in terms of spherical waves allows a direct identification of the series of spherical harmonics with the multipole expansion, to be referred to as expansion in *irreducible* multipoles [6], whereas it has to be explicitly introduced, in the case of plane waves, by expanding either the (Fourier transformed) source current density, as shown in section 2, or the source charge and current density about the retarded time, as performed in section 3, the corresponding

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multipoles being referred to as *reducible* multipoles [6]. These two sets of multipoles cannot be straightforwardly compared and some caution is needed in their practical utilization, as discussed in section 4. The results thus obtained are briefly summarized in section 5.

2. Emission by reducible multipoles on the basis of the emission formula

The time-averaged power P radiated by a source current density $\boldsymbol{J}$ can be obtained conveniently from the rate at which work is done by $\boldsymbol{J}$ itself against the electric field $\boldsymbol{E}$ it generates, namely,

$$P = -\frac{1}{T}\int_{-T/2}^{T/2} \mathrm{d}t \int \mathrm{d}^3r\, \boldsymbol{J}(\boldsymbol{r}, t)\cdot \boldsymbol{E}(\boldsymbol{r}, t) \tag{1}$$

T being an arbitrarily long time compared to periods of interest. On expressing both $\boldsymbol{J}(\boldsymbol{r}, t)$ and $\boldsymbol{E}(\boldsymbol{r}, t)$ in terms of their Fourier transforms $\boldsymbol{J}(\boldsymbol{k}, \omega)$ and $\boldsymbol{E}(\boldsymbol{k}, \omega)$, respectively, and utilizing the solution of the inhomogeneous wave equation *in vacuo*, from (1) one gets the emission formula [1, 4, 7]

$$P = \frac{1}{c^3T}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\left\langle\left|\hat{\boldsymbol{k}}\times\omega\boldsymbol{J}\left(\frac{\omega}{c}\hat{\boldsymbol{k}}, \omega\right)\right|^2\right\rangle \tag{2}$$

where the angular brackets denote the average over the angle of emission, i.e. $\langle\ldots\rangle \equiv (1/4\pi)\int \mathrm{d}^2\Omega_k(\ldots)$, $\mathrm{d}^2\Omega_k$ being the element of solid angle about the direction of emission $\hat{\boldsymbol{k}}$.

On using the definition of the Fourier transform, for which $\boldsymbol{J}((\omega/c)\hat{\boldsymbol{k}}, \omega) = \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\int \mathrm{d}^3r\, \mathrm{e}^{-\mathrm{i}(\omega/c)\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\boldsymbol{J}(\boldsymbol{r}, t)$, and expressing the exponential $\mathrm{e}^{-\mathrm{i}(\omega/c)\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}$ in terms of its series expansion, the emission formula (2) can be put in a form pertinent to the emission by reducible multipoles, i.e.

$$P = \frac{1}{c^3T}\int_{-T/2}^{T/2} \mathrm{d}t\, \langle a^2(\hat{\boldsymbol{k}}; t) - (\hat{\boldsymbol{k}}\cdot\boldsymbol{a}(\hat{\boldsymbol{k}}; t))^2\rangle \tag{3a}$$

the quantity $\boldsymbol{a}(\hat{\boldsymbol{k}}; t)$ exhibiting the multipole character of the emission process, namely,

$$\begin{aligned}\boldsymbol{a}(\hat{\boldsymbol{k}}; t) &\equiv \sum_{n=0}^{\infty}\frac{1}{n!c^n}\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\int \mathrm{d}^3r\, (\hat{\boldsymbol{k}}\cdot\boldsymbol{r})^n\boldsymbol{J}(\boldsymbol{r}, t)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!c^n}\hat{\boldsymbol{k}}^n\boldsymbol{\cdot}\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\boldsymbol{M}^{(n+1)}(t)\end{aligned} \tag{3b}$$

where the bold-face dot $\boldsymbol{\cdot}$ denotes the tensor product and amounts to saturating the indices of the n-rank dyad $\hat{\boldsymbol{k}}^n$ with the first n indices of the $(n+1)$-rank tensor

$$\boldsymbol{M}^{(n+1)}(t) \equiv \int \mathrm{d}^3r\, \boldsymbol{r}^n\boldsymbol{J}(\boldsymbol{r}, t). \tag{3c}$$

The tensor (3c), which is to be referred to as the 2^{n+1}-reducible multipole tensor, can be expressed in terms of both time derivatives of electric 2^{n+1}-moments and magnetic multipole moments ($n \geqslant 1$), for example, $\boldsymbol{M}^{(1)}(t) = \dot{\boldsymbol{d}}(t)$, $M_{ij}^{(2)}(t) = \frac{1}{2}\dot{q}_{ij}(t) + c\varepsilon_{ijk}m_k(t)$, where a dot denotes differentiation with respect to time and where $\boldsymbol{d}(t) \equiv \int \mathrm{d}^3r\, \boldsymbol{r}\rho(\boldsymbol{r}, t)$ is the electric dipole moment, $q_{ij}(t) \equiv \int \mathrm{d}^3r\, r_ir_j\rho(\boldsymbol{r}, t)$ the electric quadrupole moment and $\boldsymbol{m}(t) \equiv (1/2c)\int \mathrm{d}^3r\, \boldsymbol{r}\times\boldsymbol{J}(\boldsymbol{r}, t)$ the magnetic dipole moment ($\rho(\boldsymbol{r}, t)$ is the source charge density and ε_{ijk} is the three-dimensional (3D) anti-symmetric unit tensor).

Whereas result (3) is of immediate utilization for the multipole emission per unit solid angle, to determine the total power radiated requires the angular average. The latter amounts to calculating integrals of the form $\int \mathrm{d}^2\Omega_{\boldsymbol{k}}\, \hat{k}_{i_1}\hat{k}_{i_2}\ldots\hat{k}_{i_m}$ with m even, the average over any product of an odd number of factors $\hat{\boldsymbol{k}}$ being zero because of the spherical symmetry. Such integrals can be evaluated on saturating all pairs of indices except one, and noting that $\hat{k}_i\hat{k}_i = 1$ and $\langle\hat{k}_i\hat{k}_j\rangle = \delta_{ij}/3$. The angular average is, thus, proportional to a sum of as many products of $(m/2)$ unit tensors δ_{ij} as required for the correct symmetry with respect to the interchange of indices. The constant of proportionality can then be obtained on saturating the indices in pairs [4, 8]. Whereas the foregoing procedure is straightforward for the averages for which the number of $\hat{\boldsymbol{k}}$'s is not too high, for example $\langle\hat{k}_i\hat{k}_j\hat{k}_r\hat{k}_s\rangle = (1/15)(\delta_{ij}\delta_{rs}+\delta_{ir}\delta_{js}+\delta_{is}\delta_{jr})$, it becomes rather cumbersome for a large (even) number of factors $\hat{\boldsymbol{k}}$, with the result that it does not appear to be possible to express the right-hand side of (3*a*) in terms of series of a generic average quantity.

On account of the quadratic dependence of the power (3*a*) with respect to $\boldsymbol{a}(\hat{\boldsymbol{k}};t)$, to a given order in $(1/c^m)$, with m an odd integer greater or equal to 5, along with terms quadratic in $\boldsymbol{M}^{(n+1)}$, i.e. emission by independent multipoles, there are contributions to order $(1/c^m)$ also from cross terms, i.e. emission from the interaction of lower and higher order multipoles. Explicitly, to evaluate the power (3*a*) to order $(1/c^5)$, say, one has to consider the terms $n = 0, 1, 2$ of (3*b*), with the result that

$$P = \frac{1}{T}\int_{-T/2}^{T/2}\mathrm{d}t\left\{\frac{2}{3c^3}\dot{M}_i^{(1)}\dot{M}_i^{(1)} + \frac{1}{15c^5}[4\ddot{M}_{ij}^{(2)}\ddot{M}_{ij}^{(2)} - \ddot{M}_{ij}^{(2)}\ddot{M}_{ji}^{(2)} - (\ddot{M}_{ii}^{(2)})^2 + 2(2\,\dddot{M}_{iij}^{(3)}\,\dot{M}_j^{(1)} - \dot{M}_i^{(1)}\,\dddot{M}_{ijj}^{(3)})]\right\} \tag{4a}$$

where the sum over dummy (that appear twice) indices is implied. The first term of (4*a*) is just the well known electric dipole emission $(2/3c^3)\overline{|\ddot{\boldsymbol{d}}(t)|^2}$, the overbar denoting the average over time [4, 7, 8]. The contribution to P quadratic in $\ddot{\boldsymbol{M}}^{(2)}$ can be shown to be equal to $(1/15c^5)(\frac{3}{4}\overline{\dddot{Q}_{ij}\dddot{Q}_{ij}} + 10c^2\overline{|\ddot{m}|^2})$, with $Q_{ij} \equiv q_{ij} - \frac{1}{3}\mathrm{Tr}\{\boldsymbol{q}\}\delta_{ij}$ the traceless electric quadrupole moment; this contribution is, thus the standard electric quadrupole and magnetic dipole emission [3, 5, 7]. The contribution to (4*a*) due to the cross effect between $\boldsymbol{M}^{(3)}$ and $\boldsymbol{M}^{(1)}$, for which

$$2\,\dddot{M}_{iij}^{(3)}\,\dot{M}_j^{(1)} - \dot{M}_i^{(1)}\,\dddot{M}_{ijj}^{(3)} = \frac{1}{3}\ddot{\boldsymbol{d}}(t)\cdot\left(\frac{\mathrm{d}^4}{\mathrm{d}t^4}\int\mathrm{d}^3r\,\boldsymbol{r}r^2\rho(\boldsymbol{r},t) - 5\frac{\mathrm{d}^3}{\mathrm{d}t^3}\int\mathrm{d}^3r\,\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{J}(\boldsymbol{r},t))\right) \tag{4b}$$

originates from the interaction of the electric dipole moment with both the electric octupole moment, $\int\mathrm{d}^3r\,r_ir_jr_k\rho(\boldsymbol{r},t)$, and the magnetic quadrupole moment, $\varepsilon_{imn}\int\mathrm{d}^3r\,r_jr_mJ_n(\boldsymbol{r},t)$.

To our knowledge, result (4*b*) is new: the only terms considered hitherto for the multipole emission to order $(1/c^5)$ have been the electric quadrupole emission and the magnetic dipole emission [4, 8]. It appears somewhat surprising that such a relevant contribution to the emission has been so far overlooked, although its relevance can be easily verified by considering the specific case for which the source of emission is a point charge: only on account of contribution (4*b*) one obtains the result predicted by the Liénard formula, as shown in the appendix. On account of result (4*b*), the well known standard formula for the emission by reducible multipoles to order $(1/c^5)$, which includes only emission by independent multipoles, cf, for example, equation (9-61) of [8], needs to be rectified.

3. An exact expression for the emission in terms of reducible multipoles

On the basis of the emission formula (2) it appears quite difficult to obtain the emission to an arbitrary mutlipole order, notwithstanding the fact that the cross terms between multipoles of different order are not explicitly apparent. One can, however, put equation (2) in a form such that the angular average no longer occurs (let us recall that it is just the angular average that makes equation (2) somewhat unsuitable to treat the multipole emission). More specifically, (i) express the $(\hat{\boldsymbol{k}}\cdot\boldsymbol{J})$ part of the integrand of (2) in terms of the charge density $\rho(\boldsymbol{k},\omega)$ by means of the continuity equation, (ii) take the inverse spatial Fourier transform, which makes the integration over the solid angle straightforward and (iii) after taking the inverse temporal Fourier transform, carry out the ω-integration, which yields the function $\delta(t-t'-|\boldsymbol{r}-\boldsymbol{r}'|/c)$. The radiated power (2) can, thus, be written as [7]

$$P=\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{d}t\left\{\int\mathrm{d}^3r\int\mathrm{d}^3r'\,\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\left([\rho(\boldsymbol{r}',t')]\frac{\partial\rho(\boldsymbol{r},t)}{\partial t}-\frac{1}{c^2}[\boldsymbol{J}(\boldsymbol{r}',t')]\cdot\frac{\partial\boldsymbol{J}(\boldsymbol{r},t)}{\partial t}\right)\right\}\tag{5}$$

the square brackets meaning that the quantities within have to be evaluated at the retarded time $t'=t-|\boldsymbol{r}-\boldsymbol{r}'|/c$.

With reference to the integrand of (5), let us make an expansion about the retarded time, so that

$$[\{\rho,\boldsymbol{J}\}(\boldsymbol{r}',t')]=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{R}{c}\right)^n\frac{\partial^n\{\rho,\boldsymbol{J}\}}{\partial t^n}\bigg|_{t'=t}\qquad R\equiv|\boldsymbol{r}-\boldsymbol{r}'|.\tag{6}$$

On noting that (i) the even powers of (6) yield zero upon the time integration in (5), which amounts to replacing n by $(2n+1)$ in (6), and (ii) the $n=0$ term of the ρ part of (5) is zero, being proportional to $(\mathrm{d}/\mathrm{d}t)\int\mathrm{d}^3r\,\rho(\boldsymbol{r},t)(=0)$, so that $(2n+1)\to(2n+3)$ for the ρ part, one has

$$P=\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{d}t\left\{\sum_{n=0}^{\infty}\frac{1}{c^{2n+3}}\right.$$
$$\left.\times\int\mathrm{d}^3r\int\mathrm{d}^3r'\left(\frac{R^{2n}}{(2n+1)!}\frac{\partial^{2n+1}\boldsymbol{J}'}{\partial t^{2n+1}}\cdot\frac{\partial\boldsymbol{J}}{\partial t}-\frac{R^{2(n+1)}}{(2n+3)!}\frac{\partial^{2n+3}\rho'}{\partial t^{2n+3}}\frac{\partial\rho}{\partial t}\right)\right\}\tag{7}$$

where the prime symbol denotes the dependence on $\boldsymbol{r}'$. On making use of the continuity equation to eliminate the charge density and by suitable integrations by parts over time and space, and using the identity

$$\frac{\partial}{\partial\boldsymbol{r}'}\frac{\partial}{\partial\boldsymbol{r}}R^{2(n+1)}=-2(n+1)R^{2n}(2n\hat{\boldsymbol{R}}\hat{\boldsymbol{R}}+\boldsymbol{I})$$

$\boldsymbol{I}$ being the unit tensor, one obtains

$$P=\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{d}t\left\{\sum_{n=0}^{\infty}\frac{(-1)^n4(n+1)}{(2n+3)!c^{2n+3}}\int\mathrm{d}^3r\right.$$
$$\left.\times\int\mathrm{d}^3r'\,R^{2n}\left((n+1)\frac{\partial^{n+1}\boldsymbol{J}'}{\partial t^{n+1}}\cdot\frac{\partial^{n+1}\boldsymbol{J}}{\partial t^{n+1}}-n\left(\hat{\boldsymbol{R}}\cdot\frac{\partial^{n+1}\boldsymbol{J}'}{\partial t^{n+1}}\right)\left(\hat{\boldsymbol{R}}\cdot\frac{\partial^{n+1}\boldsymbol{J}}{\partial t^{n+1}}\right)\right)\right\}.\tag{8}$$

The $n = 0$ term of (8) is just the electric dipole emission, cf also (4a). To express the nth term of (8) in terms of the 2^{n+1}-reducible multipole tensor $\boldsymbol{M}^{(n+1)}$, cf (3c), one has to separate the space variables $\boldsymbol{r}$ and $\boldsymbol{r}'$ which are coupled through $R^{2n} = ((r^2+r'^2)-2\boldsymbol{r}\cdot\boldsymbol{r}')^n$. On applying twice Newton's binomial formula, one finally gets

$$P = \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{d}t \left\{ \frac{2}{3c^3}\dot{M}_i^{(1)}\dot{M}_i^{(1)} + \frac{4}{c^3}\sum_{n=1}^{\infty}\frac{2^n(n+1)!}{(2n+3)!c^{2n}}(A_n+B_n)\right\} \tag{9a}$$

the dipole term being written out explicitly,

$$\begin{aligned} A_n \equiv \sum_{k=0} \frac{1}{(2^k k!)^2}\Bigg\{ & \frac{n+1}{(n-2k)!}|\mathrm{Tr}^{(k)}\tilde{\boldsymbol{M}}^{(n+1)}|^2 - \frac{1}{a(n-2k-1)!}(|\mathrm{Tr}^{(k)}\tilde{\boldsymbol{M}}_{\Leftarrow}^{(n+1)}|^2 \\ & +(\mathrm{Tr}^{(k)}\tilde{\boldsymbol{M}}^{(n+1)})\cdot(\mathrm{Tr}^{(k)}\tilde{\boldsymbol{M}}^{(n+1)})^{\mathrm{T}} + 2(\mathrm{Tr}^{(k)}\tilde{\boldsymbol{M}}_{\Leftarrow}^{(n+2)})\cdot(\mathrm{Tr}^{(k)}\tilde{\boldsymbol{M}}^{(n)})) \\ & -\frac{1}{2(n-2k-2)!}(\mathrm{Tr}^{(k)}\tilde{\boldsymbol{M}}_{\Leftarrow}^{(n+1)})\cdot(\mathrm{Tr}^{(k+1)}\tilde{\boldsymbol{M}}^{(n+1)})\Bigg\} \end{aligned} \tag{9b}$$

and

$$\begin{aligned} B_n \equiv \sum_{k=0}\sum_{k'=0}^{\mathrm{int}(k/2)} \frac{1}{2^k k'!(k-k'+1)!}\Bigg\{ & \frac{(n+1)}{(n-k-1)!}(\mathrm{Tr}^{(k')}\tilde{\boldsymbol{M}}^{(n-k+2k')}) \\ & \cdot(\mathrm{Tr}^{(k-k'+1)}\tilde{\boldsymbol{M}}^{(n+k-2k'+2)}) \\ & +\frac{1}{2(n-k-2)!}((\mathrm{Tr}^{(k')}\tilde{\boldsymbol{M}}_{\Leftarrow}^{(\tilde{n}-k+2k')})\cdot(\mathrm{Tr}^{(k-k'+1)}\tilde{\boldsymbol{M}}_{\Leftarrow}^{(n+k-2k'+2)}) \\ & +(\mathrm{Tr}^{(k')}\tilde{\boldsymbol{M}}^{(n-k+2k')})\cdot(\mathrm{Tr}^{(k-k'+1)}\tilde{\boldsymbol{M}}^{(n+k-2k'+2)})^{\mathrm{T}}) \\ & +\frac{1}{4(k-k'+2)(n-k-3)!}(\mathrm{Tr}^{(k')}\tilde{\boldsymbol{M}}^{(n-k+2k'-2)})\cdot(\mathrm{Tr}^{(k-k'+2)}\tilde{\boldsymbol{M}}_{\Leftarrow}^{(n+k-2k'+4)}\Bigg\}. \end{aligned} \tag{9c}$$

In equations (9b, c)

$$\tilde{\boldsymbol{M}}^{(n+1)} \equiv \frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\int \mathrm{d}^3r\, \boldsymbol{r}^n \boldsymbol{J}(\boldsymbol{r},t)$$

which is the $(n+1)$th time derivative of the 2^{n+1}-reducible multipole tensor (3c) (the tilde over a tensor just denotes such a differentiation), and

$$\tilde{\boldsymbol{M}}_{\Leftarrow}^{(n+1)} \equiv \frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\int \mathrm{d}^3r\, \boldsymbol{r}^{n-1}(\boldsymbol{r}\cdot\boldsymbol{J}(\boldsymbol{r},t))$$

the operator $\mathrm{Tr}^{(k)}$ applied to a tensor saturates its first $2k$ indices in pairs, the symbol $\cdot$ represents the usual tensor product and the modulus square of a tensor is defined as the sum of the squares of its elements; moreover, with reference to the summation over k, there is an upper limit which is fixed by the condition that the factorial of negative numbers has to be excluded. Whereas the (infinite) summation over the integer $n(\geqslant 1)$ originates from the expansion about the retarded time, cf (6), the two (finite) summations over the integers $k(\geqslant 0)$ and $k'(\geqslant 0)$ have to do with the binomial expansion applied twice to the quantity R^{2n} occurring in (8).

Expressions (9), the main result of this paper, yield the emission by reducible multipoles to any order in $(1/c^{3+2n})$. The emission at a given multipole order, labelled by the index n, comprises terms quadratic in respect to a single multipole, cf the terms of (9b) except the framed one, as well as bilinear terms related to the interaction of pairs of multipoles of different order, cf the framed term of (9b) and the contribution (9c). The latter cross terms, in particular, so far appear to have been overlooked in the literature [4, 8]. More specifically, for $n = 1$, i.e. the contribution of order $(1/c^5)$ to the multipole emission, the relevant cross terms are given by the two framed terms of (9*b*) and (9*c*) with $n = 1$, $k = k' = 0$, and equations (9) just reproduce the result (4*a*), with no contribution from both the last term of (9*b*) and the terms of (9*c*) other than the framed one. The $n = 2$ contribution out of (9), i.e. the order $(1/c^7)$-multipole emission, is given explicitly in table 1, for which no contribution comes from the last term of (9*c*), all the other terms giving a $k = 0$ contribution, the first term of both (9*b*) and (9*c*) yielding also a contribution for $k = 1$. It is to be noted that the ability to obtain the exact result (9) relies on having utilized the expression (5) for the radiated power; the utilization of the (apparently more inspiring) emission formula (2), instead, does not appear to make it possible to achieve such a result, due to the awkward angular average one has to deal with, as discussed in section 2.

Table 1. Explicit expression of A_n and B_n for $n = 2$, cf equations (9*b*, *c*).

A_n	$\frac{3}{2}\tilde{M}^{(3)}_{ijk}\tilde{M}^{(3)}_{ijk} - \frac{1}{2}\tilde{M}^{(3)}_{ijj}\tilde{M}^{(3)}_{ikk} - \frac{1}{2}\tilde{M}^{(3)}_{ijk}\tilde{M}^{(3)}_{kji} - \tilde{M}^{(2)}_{ij}\tilde{M}^{(4)}_{ijkk} - \frac{1}{2}\tilde{M}^{(3)}_{iij}\tilde{M}^{(3)}_{jkk} + \frac{3}{4}\tilde{M}^{(3)}_{iij}\tilde{M}^{(3)}_{kkj}$
B_n	$3\tilde{M}^{(2)}_{ij}\tilde{M}^{(4)}_{kkij} + \frac{1}{2}\tilde{M}^{(2)}_{ii}\tilde{M}^{(4)}_{jjkk} + \frac{1}{2}\tilde{M}^{(2)}_{ij}\tilde{M}^{(4)}_{kkji} + \frac{3}{4}\tilde{M}^{(1)}_{i}\tilde{M}^{(5)}_{jjkki}$

4. Reducible multipoles versus irreducible multipoles

A thorough treatment of multipole emission *in vacuo* has been given on the basis of the expansion of the electromagnetic field in terms of vector spherical waves, by which the emission in terms of irreducible multipoles is obtained [1, 2]. For a comparative discussion of our results (4) and (9), expressed in terms of reducible multipoles, let us first recall briefly the basic points of the treatment of [1] and [2], extending it to non-monochromatic sources. Instead of using (1), on evaluating the (time-averaged) radiated power in terms of the (time-averaged) flux of the Poynting vector through an arbitrarily large spherical surface enclosing the radiation sources, one has

$$P = \frac{c}{4\pi T}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\int \mathrm{d}^2\Omega\, r^2 |\boldsymbol{E}(\boldsymbol{r},\omega)|^2 \tag{10a}$$

$$= \frac{c^3}{4\pi T}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\sum_{l,m}\left(\left|\frac{a_E(l,m;\omega)}{\omega}\right|^2 + \left|\frac{a_M(l,m;\omega)}{\omega}\right|^2\right) \tag{10b}$$

with

$$\boldsymbol{E}(\boldsymbol{r},\omega) = \frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}(-\mathrm{i})^{l+1}(-a_E(l,m;\omega)\hat{\boldsymbol{r}}\times\boldsymbol{X}_{lm} + a_M(l,m;\omega)\boldsymbol{X}_{lm}) \qquad k \equiv \omega/c \tag{11}$$

for the (time-Fourier-transformed) electric field in the radiation zone, cf, for example, equation (16.73) of [1], where $\boldsymbol{X}_{lm}(\theta,\phi) \equiv -(\mathrm{i}/\sqrt{l(l+1)})\boldsymbol{r}\times\nabla Y_{lm}(\theta,\phi)$ are vector spherical functions, with $Y_{lm}(\theta,\phi)$ spherical harmonics, the orthogonality properties of which are exploited to carry out the integration over the solid angle in (10*a*). The coefficients a_E and a_M are given by, cf, for example, equations (16.91) and (16.92) of [1],

$$\begin{pmatrix} a_E(l,m;\omega) \\ a_M(l,m;\omega)\end{pmatrix} = \frac{4\pi k^2}{\mathrm{i}\sqrt{l(l+1)}}\int \mathrm{d}^3r\, Y^*_{lm}\begin{pmatrix}\rho(\boldsymbol{r},\omega)\frac{\partial}{\partial r}(rj_l(kr)) + \frac{\mathrm{i}k}{c}(\boldsymbol{r}\cdot\boldsymbol{J}(\boldsymbol{r},\omega))\,j_l(kr) \\ \nabla\cdot\left(\frac{\boldsymbol{r}\times\boldsymbol{J}(\boldsymbol{r},\omega)}{c}\right)j_l(kr)\end{pmatrix} \tag{12}$$

where $j_l(x)$ are spherical Bessel functions of the first kind; $\rho(\boldsymbol{r},\omega)$ and $\boldsymbol{J}(\boldsymbol{r},\omega)$ are the (time-Fourier-transformed) radiation sources. For the specific case of monochromatic sources, for which, for example, $\rho(\boldsymbol{r},\omega)=\pi\rho(\boldsymbol{r})(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$, equation (10$b$) reduces to Jackson's equation (16.79) [1].

With reference to equation (10b), let us (i) use the definition of the time Fourier transform of the radiation sources, for example, $\rho(\boldsymbol{r},\omega)=\int \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\rho(\boldsymbol{r},t)$, (ii) express the Bessel function $j_l(kr)$, $k\equiv\omega/c$, in terms of its series representation, i.e. [9],

$$j_l\left(\frac{\omega r}{c}\right)=\left(\frac{\omega r}{c}\right)^l\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!(2(n+l)+1)!!}\left(\frac{\omega r}{c}\right)^{2n} \tag{13}$$

(iii) integrate by parts with respect to time t and t', such that, for example

$$\int\frac{\mathrm{d}\omega}{2\pi}\int\mathrm{d}t\int\mathrm{d}t'\,\omega^{2(n+n'+m)}\,\mathrm{e}^{\mathrm{i}\omega(t-t')}\rho(\boldsymbol{r},t)\rho(\boldsymbol{r}',t')$$
$$=\int\mathrm{d}t\int\mathrm{d}t'\left((-1)^n\frac{\partial^{2n+m}}{\partial t^{2n+m}}\rho(\boldsymbol{r},t)\right)\left((-1)^{n'}\frac{\partial^{2n'+m}}{\partial t^{2n'+m}}\rho(\boldsymbol{r}',t')\right)\int\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{\mathrm{i}\omega(t-t')}$$

the ω-integration now reducing simply to $\delta(t-t')$, which makes one of the integrations over time straightforward. With these steps, expression (10b) for the radiated power takes on the form

$$P=\frac{1}{4\pi cT}\int_{-T/2}^{T/2}\mathrm{d}t\,\sum_{l,m}(|a_E(l,m;t)|^2+a_M(l,m;t)|^2) \tag{14a}$$

with, from (12),

$$\begin{pmatrix}a_E(l,m;t)\\ a_M(l,m;t)\end{pmatrix}=\frac{4\pi}{\mathrm{i}\sqrt{l(l+1)}c^l}\sum_{n=0}^{\infty}\frac{2n+l+1}{2^n n!(2(n+l)+1)!!c^{2n}}$$
$$\times\frac{\mathrm{d}^{2n+l+1}}{\mathrm{d}t^{2n+l+1}}\begin{pmatrix}Q_{lm}^{(2n)}(t)-\tilde{M}_{lm}^{(2n)}(t)\\ -M_{lm}^{(2n)}(t)\end{pmatrix} \tag{14b}$$

where

$$\begin{pmatrix}Q_{lm}^{(2n)}\\ \tilde{M}_{lm}^{(2n)}\\ -M_{lm}^{(2n)}\end{pmatrix}=\int\mathrm{d}^3r\,r^{2n+l}Y_{lm}^*\begin{pmatrix}\rho(\boldsymbol{r},t)\\ \frac{1}{2n+l+1}\frac{\boldsymbol{r}\cdot\dot{\boldsymbol{J}}(\boldsymbol{r},t)}{c^2}\\ \frac{1}{2n+l+1}\nabla\cdot\frac{(\boldsymbol{r}\times\boldsymbol{J}(\boldsymbol{r},t))}{c}\end{pmatrix}. \tag{14c}$$

With respect to result (9), notwithstanding the formal difference of the respective multipoles, namely, multipole moments in the form of Cartesian reducible tensors in (9) compared to multipole moments in terms of irreducible tensors, cf (14c), it appears that, because a_E and a_M of (14a) are both proportional to $(1/c^{l+2n})$, cf (14b), to evaluate the multipole contribution at a selected order in $(1/c)$ one has to collect the relevant terms originating from the infinite summations over l and n, the latter one being squared, which makes the practical utilization of result (14) significantly less straightforward than (9).

With reference specifically to Jackson's book [1], result (14) constitutes a generalization of Jackson's result (16.79) inasmuch as it accounts for contributions from non-monochromatic multipoles to any order in $(1/c^{2n})$. As for Jackson's approximated expression (16.93)–(16.94) for the electric multipole coefficient a_E, valid in the long-wavelength limit, it is obtained from Jackson's (16.91) by neglecting the $(\boldsymbol{r}\cdot\boldsymbol{J})$ term with respect to the ρ-term, the former one being of higher order than the latter one [10], as apparent in (14c). Such an approximation, valid within a well-defined n-value, no longer holds on summing over n, since, for example, the ρ term for $n = 1$ is just of the same order as the $(\boldsymbol{r} \cdot \boldsymbol{J})$ term for $n = 0$, cf (14*b*). As a consequence, to order $(1/c^5)$, in addition to the $(l = 2,\ n = 0)$ contribution, there is also the contribution resulting from the $(l = 1)$ term combined with the product of the $n = 0$ and $n = 1$ terms from the square of the summation over n, which just leads to result (4*a*); the latter contribution, along with all the other cross terms of higher order, is (inconsistently) disregarded in Jackson's result (16.93)–(16.94), as well as in the corresponding ones of [2] and [10]. In this regard, it is worth noting a private communication by B French to Blatt and Weisskopf, cf footnote on p 806 of [2], where he remarked that it is somewhat surprising that a quantity which depends on $\boldsymbol{J}$ only through its curl, cf, for example, Jackson's equation (16.89), may be put in a form, although approximated, which depends on $\boldsymbol{J}$ only through its divergence, and hence only on ρ. In fact, result (14) just contains, along with contributions from the ρ term, also contributions from the $(\boldsymbol{r} \cdot \boldsymbol{J})$ term.

5. Summary

On the basis of the emission formula (2), the radiated power has been obtained in terms of reducible multipoles up to order $(1/c^5)$, cf equations (4); such a result rectifies the standard formula for the multipoles emission, for which the contribution (4*b*) is not accounted for. More generally, on making use of expression (5) for the (time-averaged) radiated power in terms of the space and retarded time-dependent sources, an exact expression for the emission by reducible multipoles has been obtained, cf equations (9), which includes, for the first time, the contribution from the interaction between multipoles of different order. On the other hand, such an exact result is hardly achievable on utilizing the emission formula (2), for which the radiation source is simply the (Fourier-transformed) current density, due to the awkwardness of the angular average one has to deal with. The results thus obtained have been discussed in respect to the treatment of the multipole emission in terms of irreducible tensors: it is just the consideration of the interaction terms between different reducible multipoles that makes the two approached equivalent, as expected.

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Appendix. Multipole emission by a pointlike charge

In the specific case of a point charge, for which $\rho(\boldsymbol{r}, t) = q\delta(\boldsymbol{r} - \boldsymbol{r}_q(t))$ and $\boldsymbol{j}(\boldsymbol{r}, t) = q\delta(\boldsymbol{r} - \boldsymbol{r}_q(t))\dot{\boldsymbol{r}}_q(t)$, $\boldsymbol{r}_q(t)$ being the charge's instantaneous position, the quantity occurring under the time average in (4*a*), without the cross term (4*b*), can be shown to be

$$P^{(\mathrm{pc})}(t) = \frac{2q^2}{3c^3}|\ddot{\boldsymbol{r}}_q|^2 + \frac{2q^2}{15c^5}\{8|\dot{\boldsymbol{r}}_q|^2|\ddot{\boldsymbol{r}}_q|^2 + (\dot{\boldsymbol{r}}_q \cdot \ddot{\boldsymbol{r}}_q)^2\} + P(t; \boldsymbol{r}_q) + \mathrm{O}(c^{-7}) \tag{A.1a}$$

$$\begin{aligned} P(t; \boldsymbol{r}_q) \equiv \frac{2q^2}{15c^5}\{&2|\boldsymbol{r}_q|^2|\dddot{\boldsymbol{r}}_q|^2 + 7(\boldsymbol{r}_q \cdot \dot{\boldsymbol{r}}_q)(\ddot{\boldsymbol{r}}_q \cdot \dddot{\boldsymbol{r}}_q) + 2(\boldsymbol{r}_q \cdot \ddot{\boldsymbol{r}}_q)(\dot{\boldsymbol{r}}_q \cdot \dddot{\boldsymbol{r}}_q) - (\boldsymbol{r}_q \cdot \dddot{\boldsymbol{r}}_q)^2 \\ &-3(\boldsymbol{r}_q \cdot \dddot{\boldsymbol{r}}_q)(\dot{\boldsymbol{r}}_q \cdot \ddot{\boldsymbol{r}}_q)\}. \end{aligned} \tag{A.1b}$$

On the other hand, the instantaneous power radiated by a point charge is given by the well known Liénard formula [1, 8]

$$P^{(\mathrm{L})}(t) = \frac{2q^2}{3c^3}\gamma^6\left(|\ddot{\boldsymbol{r}}_q|^2 - \frac{|\dot{\boldsymbol{r}}_q \times \ddot{\boldsymbol{r}}_q|^2}{c^2}\right)$$
$$= \frac{2q^2}{3c^3}|\ddot{\boldsymbol{r}}_q|^2 + \frac{2q^2}{3c^5}\{2|\dot{\boldsymbol{r}}_q|^2|\ddot{\boldsymbol{r}}_q|^2 + (\dot{\boldsymbol{r}}_q \cdot \ddot{\boldsymbol{r}}_q)^2\} + \mathrm{O}\left(\frac{1}{c^7}\right) \qquad \text{(A.2)}$$

γ being the relativistic Lorentz factor, whose expansion to order $(1/c^2)$ yields the second expression of (A.2).

Except for the electric dipole term, expressions (A.1) and (A.2) are manifestly different, and such a difference persists even on time average, since (A.1*b*) in no way can be put, via integrations by parts over time, in a form independent of the charge's position $\boldsymbol{r}_q$, as it would be required to reconcile (A.1) with (A.2). The agreement with (A.2) can be only achieved on considering the cross term (4*b*), whose contribution to the 'instantaneous' power radiated by a point charge is

$$P^{(\mathrm{cross})}(t) = -\frac{4q^2}{15c^5}|\dot{\boldsymbol{r}}_q|^2(\dot{\boldsymbol{r}}_q \cdot \dddot{\boldsymbol{r}}_q) - P(t; \boldsymbol{r}_q). \qquad \text{(A.3)}$$

Adding up (A.1) and (A.3) yields

$$P^{(\mathrm{pc})}(t) + P^{(\mathrm{cross})}(t) = \frac{2q^2}{3c^3}|\ddot{\boldsymbol{r}}_q|^2 + \frac{2q^2}{15c^5}\{8|\dot{\boldsymbol{r}}_q|^2|\ddot{\boldsymbol{r}}_q|^2 + (\dot{\boldsymbol{r}}_q \cdot \ddot{\boldsymbol{r}}_q)^2 - 2|\dot{\boldsymbol{r}}_q|^2(\dot{\boldsymbol{r}}_q \cdot \dddot{\boldsymbol{r}}_q)\} \qquad \text{(A.4)}$$

which is just equal to (A.2), on time average and by virtue of the identity

$$\int \mathrm{d}t\; |\dot{\boldsymbol{r}}_q|^2(\dot{\boldsymbol{r}}_q \cdot \dddot{\boldsymbol{r}}_q) = -\int \mathrm{d}t\; \{|\dot{\boldsymbol{r}}_q|^2|\ddot{\boldsymbol{r}}_q|^2 + 2(\dot{\boldsymbol{r}}_q \cdot \ddot{\boldsymbol{r}}_q)^2\}$$

valid for periodic motions of the charge.

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